Solutions to Exam Fall Semester 2019/2020

Problem 1: Wheatstone bridge linearization [10pt]

The circuit in Fig. 1 is a linearized bridge, with $R+dR = R(1+d) = R(1+\alpha T)$, where T [°C] is the temperature and $\alpha = 0.004\,°\text{C}^{-1}$ is the temperature coefficient.

1.1 DC analysis [4pt]

- derive the output voltage as a function of the sensor resistance change dR;

Let us write the current summation at nodes A and B (which are grounded), respectively:

$$\begin{cases} \dfrac{V_c}{R} + \dfrac{V_{ref}}{R} + \dfrac{V_{out}}{R_f} = 0; \\ \dfrac{V_c}{R+dR} + \dfrac{V_{ref}}{R} = 0; \end{cases} \tag{1}$$

leading to:

$$V_{out} = \frac{R_f \cdot dR}{R^2} V_{ref}. \tag{2}$$

- show the effects of the op-amp A_1 leakage currents I_p and I_m and of the two op-amps offset voltages at the output (assume the leakage currents of the op-amp A_2 are negligible).

Let us use the superposition principle in order to evaluate the effects of the leakage currents and the offsets voltages, referring to Fig. 1.

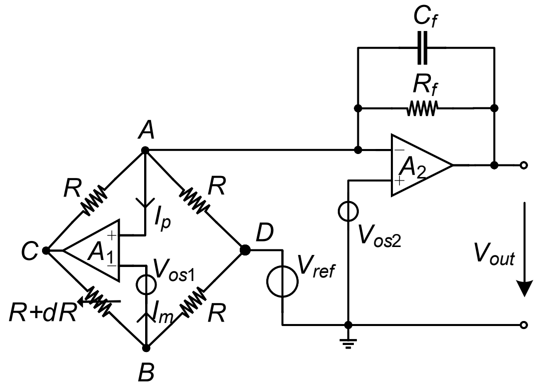

Figure 1: Linearized Wheatstone bridge. DC analysis set-up

Effect of the leakage currents I_p and I_m.

Let us write the current summation at nodes A and B (which are grounded) considering the leakage currents, respectively:

$$\begin{cases} \dfrac{V_c}{R} + \dfrac{V_{ref}}{R} + \dfrac{V_{out}}{R_f} - I_p = 0; \\ \dfrac{V_c}{R+dR} + \dfrac{V_{ref}}{R} - I_m = 0; \end{cases} \tag{3}$$

leading to:

$$V_{out} = (I_p - I_m) \cdot R_f + \frac{R_f \cdot dR}{R^2}(V_{ref} - I_m \cdot R). \tag{4}$$

Effect of A_1 offset voltage V_{os1}.

Let us write the current summation at nodes A and B considering the effect of A_1 offset voltage V_{os1}:

$$\begin{cases} \dfrac{V_c}{R} + \dfrac{V_{ref}}{R} + \dfrac{V_{out}}{R_f} = 0; \\ \dfrac{V_c - V_{os1}}{R+dR} + \dfrac{V_{ref} - V_{os1}}{R} = 0; \end{cases} \tag{5}$$

leading to:

$$V_{out} = -2\frac{V_{os1} \cdot R_f}{R} + \frac{R_f \cdot dR}{R^2}(V_{ref} - V_{os1}). \tag{6}$$

Effect of A_2 offset voltage V_{os2}.

Let us write the current summation at nodes A and B considering the effect of A_2 offset voltage V_{os2}:

$$\begin{cases} \dfrac{V_c - V_{os2}}{R} + \dfrac{V_{ref} - V_{os2}}{R} + \dfrac{V_{out} - V_{os2}}{R_f} = 0; \\ \dfrac{V_c - V_{os2}}{R+dR} + \dfrac{V_{ref} - V_{os2}}{R} = 0; \end{cases} \tag{7}$$

leading to:

$$V_{out} = \left(1 + \frac{dR \cdot R_f}{R^2}\right) V_{os2} + \frac{R_f \cdot dR}{R^2} V_{ref}. \tag{8}$$

- rewrite the previous result in terms of output offset voltage (V_{off}) and relative gain error (ε), i.e. $V_{out} = V_{off} + dR \cdot G \cdot (1-\varepsilon)$, where G is the gain in the ideal case (when $V_{off} = 0\text{V}$ and $\varepsilon = 0$).

Effects combination.

Summing up all of the previous effects leads to:

$$V_{out} = \underbrace{\frac{(I_p - I_m) \cdot R \cdot R_f + V_{os2} \cdot R - 2V_{os2} \cdot Rf}{R}}_{\text{offset error } V_{off}} + \frac{R_f \cdot dR}{R^2} V_{ref} \left(1 - \underbrace{\frac{I_m \cdot R + V_{os1} + V_{os2}}{V_{ref}}}_{\text{gain relative error } \varepsilon}\right). \tag{9}$$

1.2 Noise analysis [4pt]

Assume $dR = 0$, the amplifier A_2 to be noiseless and the amplifier A_1 having no $1/f$ noise and only an input-referred thermal noise PSD given by $4kTR_A$.

For each of the noise sources calculate:

- the transfer function from the noise source to the output;
- the power spectral density and
- the output noise voltage variance.

For this calculation, V_{ref} is set to ground. Let us define $Z = C_f \parallel R_f$ and let us identify the 4 resistors in the bridge according to the names of their terminal nodes, i.e. R_{AC}, R_{AD}, R_{CB} and R_{BD}. Let us model the noise sources of the amplifier A_1 and the resistors in the bridge as a voltage source and the noise source of R_f as a current source as shown inFig. 2.

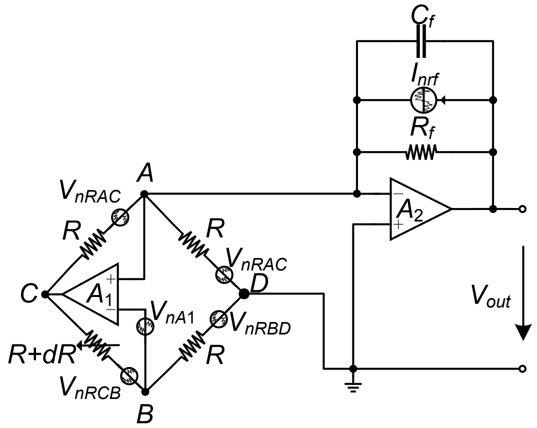

Figure 2: Linearized Wheatstone bridge. Noise analysis set-up

Let us use the superposition principle in order to evaluate the noise contributions.

Contribution of R_{AC} noise source $V_{n,RAC}$

Let us write the current summation at nodes A and B considering the noise source $V_{n,RAC}$. In this case, the node A, D and B (due to the virtual ground principle) are grounded:

$$\begin{cases} \frac{V_c - V_{n,RAC}}{R} + \frac{V_{out}}{Z} = 0; \\ \frac{V_c}{R} = 0; \end{cases} \tag{10}$$

leading to:

$$H_{n,RAC}(s) \triangleq V_{n,out} = \frac{Z}{R} V_{n,RAC} = \frac{R_f}{R} \frac{1}{1 + sC_f R_f} V_{n,RAC}; \tag{11}$$

which is a first order low-pass filters transfer function with gain R_f/R and $\omega_c = 1/(R_f C_f)$, therefore the noise equivalent-bandwidth is:

$$\Delta f = \frac{1}{4R_f C_f}; \tag{12}$$

the output noise variance is:

$$\overline{V_{out,RAC}^2} = (4kTR) \cdot \left(\frac{R_f}{R}\right)^2 \cdot \frac{1}{4R_f C_f} = \frac{kT}{C_f} \cdot \frac{R_f}{R}; \tag{13}$$

where T is the absolute temperature, k is the Boltzmann constant and $(4kTR)$ is the resistor PSD.

Contribution of R_{AD} noise source $V_{n,RAD}$

Let us write the current summation at nodes A and B considering the noise source $V_{n,RAD}$. In this case, the node A, D and B (due to the virtual ground principle) are grounded:

$$\begin{cases} \frac{V_c}{R} - \frac{V_{n,RAD}}{R} + \frac{V_{out}}{Z} = 0; \\ \frac{V_c}{R} = 0; \end{cases} \tag{14}$$

leading to:

$$H_{n,RAD}(s) \triangleq V_{n,out} = \frac{Z}{R} V_{n,RAD} = \frac{R_f}{R} \frac{1}{1 + sC_f R_f} V_{n,RAD}. \tag{15}$$

The power spectral density is given by:

$$S_{V_n,RAD}(f) = |H_{n,RAD}(s)|^2 4kTR \tag{16}$$

$H_{n,RAD}(s)$ is a first order low-pass filters transfer function with gain R_f/R and $\omega_c = 1/(R_f C_f)$, therefore the noise equivalent-bandwidth is:

$$\Delta f = \frac{1}{4R_f C_f}; \tag{17}$$

the output noise variance is:

$$\overline{V_{out,RAD}^2} = (4kTR) \cdot \left(\frac{R_f}{R}\right)^2 \cdot \frac{1}{4R_f C_f} = \frac{kT}{C_f} \cdot \frac{R_f}{R}; \tag{18}$$

where T is the absolute temperature, k is the Boltzmann constant and $(4kTR)$ is the resistor PSD.

Contribution of R_{CB} noise source $V_{n,RCB}$

Let us write the current summation at nodes A and B considering the noise source $V_{n,RCB}$. In this case, the node A, D and B (due to the virtual ground principle) are grounded:

$$\begin{cases} \frac{V_c}{R} + \frac{V_{out}}{Z} = 0; \\ \frac{V_c - V_{n,RCB}}{R} = 0; \end{cases} \tag{19}$$

leading to:

$$H_{n,RCB}(s) \triangleq V_{n,out} = \frac{Z}{R} V_{n,RCB} = \frac{R_f}{R} \frac{1}{1+sC_f R_f} V_{n,RCB}. \tag{20}$$

The power spectral density is given by:

$$S_{V_n,RCB}(f) = |H_{n,RCB}(s)|^2 4kTR \tag{21}$$

$H_{n,RCB}(s)$ is a first order low-pass filters transfer function with gain R_f/R and $\omega_c = 1/(R_f C_f)$, therefore the noise equivalent-bandwidth is:

$$\Delta f = \frac{1}{4R_f C_f}; \tag{22}$$

the output noise variance is:

$$\overline{V_{out,RCB}^2} = (4kTR) \cdot \left(\frac{R_f}{R}\right)^2 \cdot \frac{1}{4R_f C_f} = \frac{kT}{C_f} \cdot \frac{R_f}{R}; \tag{23}$$

where T is the absolute temperature, k is the Boltzmann constant and $(4kTR)$ is the resistor PSD.

Contribution of R_{BD} noise source $V_{n,RBD}$

Let us write the current summation at nodes A and B considering the noise source $V_{n,RBD}$. In this case, the node A, D and B (due to the virtual ground principle) are grounded:

$$\begin{cases} \frac{V_c}{R} + \frac{V_{out}}{Z} = 0; \\ \frac{V_c}{R} + \frac{V_{n,RBD}}{R} = 0; \end{cases} \tag{24}$$

leading to:

$$H_{n,RBD}(s) \triangleq V_{n,out} = \frac{Z}{R} V_{n,RBD} = \frac{R_f}{R} \frac{1}{1+sC_f R_f} V_{n,RBD}. \tag{25}$$

The power spectral density is given by:

$$S_{V_n,RBD}(f) = |H_{n,RBD}(s)|^2 4kTR \tag{26}$$

$H_{n,RBD}(s)$ is a first order low-pass filters transfer function with gain R_f/R and $\omega_c = 1/(R_f C_f)$, therefore the noise equivalent-bandwidth is:

$$\Delta f = \frac{1}{4R_f C_f}; \tag{27}$$

the output noise variance is:

$$\overline{V_{out,RBD}^2} = (4kTR) \cdot \left(\frac{R_f}{R}\right)^2 \cdot \frac{1}{4R_f C_f} = \frac{kT}{C_f} \cdot \frac{R_f}{R}; \tag{28}$$

where T is the absolute temperature, k is the Boltzmann constant and $(4kTR)$ is the resistor PSD.

Contribution of A_1 noise source $V_{n,A1}$

Let us write the current summation at nodes A and B considering the noise source $V_{n,A1}$. In this case, the node A and D are grounded:

$$\begin{cases} \frac{V_c}{R} + \frac{V_{out}}{Z} = 0; \\ \frac{V_c - V_{n,A1}}{R} - \frac{V_{n,A1}}{R} = 0; \end{cases} \tag{29}$$

leading to:

$$H_{n,A1}(s) \triangleq V_{n,out} = -2\frac{Z}{R} V_{n,A1} = -2\frac{R_f}{R}\frac{1}{1+sC_f R_f} V_{n,A1}. \tag{30}$$

The power spectral density is given by:

$$S_{V_n,A1}(f) = |H_{n,A1}(s)|^2 4kTR_A \tag{31}$$

$H_{n,A1}(s)$ is a first order low-pass filters transfer function with gain $-2R_f/R$ and $\omega_c = 1/(R_f C_f)$, therefore the noise equivalent-bandwidth is:

$$\Delta f = \frac{1}{4R_f C_f}; \tag{32}$$

the output noise variance is:

$$\overline{V^2_{out,A1}} = (4kTR_A) \cdot \left(-2\frac{R_f}{R}\right)^2 \cdot \frac{1}{4R_f C_f} = \frac{kT}{C_f} 4 \frac{R_A R_f}{R^2}; \tag{33}$$

where T is the absolute temperature, k is the Boltzmann constant and $(4kTR_A)$ is the amplifier PSD.

Contribution of R_f noise source $I_{n,Rf}$

Let us write the current summation at nodes A and B considering the noise source $V_{n,RBD}$. In this case, the node A, D and B (due to the virtual ground principle) are grounded:

$$\begin{cases} \frac{V_c}{R} + \frac{V_{out}}{Z} - I_{n,Rf} = 0; \\ \frac{V_c}{R} = 0; \end{cases} \tag{34}$$

leading to:

$$H_{n,Rf}(s) \triangleq V_{n,out} = I_{n,Rf} \cdot Z = \frac{R_f}{1+sC_f R_f} I_{n,Rf}. \tag{35}$$

By replacing $I_{n,Rf} = V_{n,Rf}/R_f$:

$$H_{n,Rf}(s) = \frac{1}{1+sCR_f} V_{n,Rf}. \tag{36}$$

The power spectral density is given by:

$$S_{V_n,Rf}(f) = \left|H_{n,A1}(s)\right|^2 4kTR_f \tag{37}$$

$H_{n,Rf}(s)$ is a first order low-pass filters transfer function with gain 1 and $\omega_c = 1/(R_f C_f)$, therefore the noise equivalent-bandwidth is:

$$\Delta f = \frac{1}{4R_f C_f}; \tag{38}$$

the output noise variance is:

$$\overline{V^2_{out,Rf}} = (4kTR_f)\cdot\frac{1}{4R_f C_f} = \frac{kT}{C_f}; \tag{39}$$

where T is the absolute temperature, k is the Boltzmann constant and $(4kTR_f)$ is the resistor PSD.

Finally,

- calculate the total output noise voltage variance.

By summing up the previous noise variance contributions:

$$\overline{V^2_{out,n}} = \frac{kT}{C_f}\left(4\frac{R_f}{R} + 4\frac{R_f R_A}{R^2} + 1\right) \tag{40}$$

1.3 Dimensioning the system [4pt]

Assuming ideal op-amps and $V_{REF} = 1\,\text{V}$ calculate R, R_f and C_f to achieve:

- an output temperature-to-voltage gain G_T of $0.2\,\text{V}/°\text{C}$ near $0\,°\text{C}$;
- a cut-off frequency $f_c = 1\,\text{kHz}$ and
- an output noise voltage variance of $100\,\mu\text{V}_{\text{RMS}}$ at $T = 0°\text{C} = 273\,\text{K}$, knowing that $k = 1.3807\times10^{-23}\,\text{J/K}$.

The design specifications can be translated in the following system with three equations and three unknowns (R, R_f and C_f):

$$\begin{cases} G_T = \dfrac{R_f\cdot V_{ref}\cdot\alpha}{R} = 0.2\,\text{V}/°\text{C} \\ f_c = \dfrac{1}{2\cdot\pi\cdot R_f\cdot C_f} = 1\,\text{kHz} \\ \overline{V^2_{out,n}} = \frac{kT}{C_f}\left(4\frac{R_f}{R}+1\right) = (100\,\mu\text{V}_{\text{RMS}})^2; \end{cases} \tag{41}$$

From the first equation we get that:

$$\frac{Rf}{R} = \frac{G_T}{V_{ref}\cdot\alpha} = 50. \tag{42}$$

By replacing this quantity into the third equation, the value of C_f can be calculated:

$$C_f = \frac{kT}{V_{out,n}^2}\left(4\frac{R_f}{R}+1\right) = 76\,\mathrm{pF}. \tag{43}$$

Then C_f can be replaced into the second equation to easily get $R_f = 21\,\mathrm{M\Omega}$. Finally, $R = R_f/50 = 420\,\mathrm{k\Omega}$

To summarize: $C_f = 76\,\mathrm{pF}$, $R_f = 21\,\mathrm{M\Omega}$ and $R = 420\,\mathrm{k\Omega}$.